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Algebraic Bethe ansatz solutions for the $sl(2|1)^{(2)}$ and $osp(2|1)$ models with boundary terms

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Abstract

This work is concerned with the formulation of the graded quantum inverse scattering method for a class of lattice models with reflecting boundary conditions. The $sl(2|1)^{(2)}$ and $osp(2|1)$ models are considered with their diagonal reflections in BFB grading. This allowed us to derive the eigenvalues and eigenvectors for the corresponding transfer matrices as well as explicit expressions for the Bethe ansatz equations.

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1. Introduction

Integrable quantum systems containing Fermi fields have been attracting increasing interest due to their potential applications in condensed matter physics. The prototypical examples of such systems are the supersymmetric generalizations of the Hubbard and t–J models [1–5]. They lead to a generalization of the Yang–Baxter (YB) equation [6] associated with the introduction of the a Z_2 grading [7]. These models are often derived from supersymmetric solutions of the Yang–Baxter equation invariant by the $gl(2|1)$ and $osp(2|2)$ Lie superalgebras [8–12].

In addition to the grading, it is also important to introduce open boundary conditions to study the boundary effects on the bulk system. The boundary effects can be described by scattering matrices satisfying the so-called reflection equation [13]. A systematic approach to construct such models has been developed by Sklyanin [14] who has investigated the six-vertex model with boundaries. Subsequently, this scheme has been generalized to handle a rather general class of models based on Lie algebras [15–18].

The most powerful method in the analysis of integrable models is the Bethe ansatz (BA). The algebraic BA, also known as the quantum inverse scattering method (QISM) [20], is an elegant and important generalization of the coordinate BA [21]. A successful example of the algebraic BA with boundaries is the supersymmetric free-parameter model constructed from the R-matrix associated to the four-dimensional representation of $gl(2|1)$ [22–24].

In this paper we will study two graded three-state 19-vertex models $sl(2|1)^{(2)}$ and $osp(2|1)$ with reflecting boundary conditions. Their boundary algebraic BA are delineated based in the recent progresses [25, 26], for the formulation of the open algebraic BA for their non-graded version, the Zamolodchikov–Fateev model (or $B_1^{(1)}$ model) [27] and for the Izergin–Korepin model (or $A_2^{(2)}$ model) [28], respectively [29].

This paper is organized as follows: in section 2 we define the models to be studied. In section 3, we present our detailed calculations common to both models and in section 4 the eigenspectra and the corresponding Bethe equations are explicitly presented for each model. Section 5 is reserved for conclusions.

2. The models

Let $V = V_0 \oplus V_1$ be a Z_2-graded vector space where 0 and 1 denote the even and odd parts respectively. Multiplication rules in the graded tensor product space $V \overset{s}{\otimes} V$ differ from the ordinary ones by the appearance of additional signs. The components of a linear operator $A \overset{s}{\otimes} B \in V \overset{s}{\otimes} V$ result in matrix elements of the form

$$(A \overset{s}{\otimes} B)_{\alpha\beta}^{\gamma\delta} = (-)^{p(\beta)(p(\alpha)+p(\gamma))} A_{\alpha\gamma} B_{\beta\delta}. \tag{2.1}$$

The action of the graded permutation operator $\mathcal{P}$ on the vector $|\alpha\rangle \overset{s}{\otimes} |\beta\rangle \in V \overset{s}{\otimes} V$ is defined by

$$\mathcal{P}\, |\alpha\rangle \overset{s}{\otimes} |\beta\rangle = (-)^{p(\alpha)p(\beta)}\, |\beta\rangle \overset{s}{\otimes} |\alpha\rangle \Longrightarrow (\mathcal{P})_{\alpha\beta}^{\gamma\delta} = (-)^{p(\alpha)p(\beta)} \delta_{\alpha\delta}\delta_{\beta\gamma}. \tag{2.2}$$

The graded transposition st and the graded trace str are defined by

$$\left(A^{\mathrm{st}}\right)_{\alpha\beta} = (-)^{(p(\alpha)+1)p(\beta)} A_{\beta\alpha}, \qquad \operatorname{str} A = \sum_{\alpha} (-)^{p(\alpha)} A_{\alpha\alpha}, \tag{2.3}$$

where $p(\alpha) = 1(0)$ if $|\alpha\rangle$ is an odd (even) element.

For the graded case the YB equation

$$\mathcal{R}_{12}(u)\mathcal{R}_{13}(u+v)\mathcal{R}_{23}(v) = \mathcal{R}_{23}(v)\mathcal{R}_{13}(u+v)\mathcal{R}_{12}(u) \tag{2.4}$$

and the reflection equation [14, 15]

$$\mathcal{R}_{12}(u-v)K_1^-(u)\mathcal{R}_{21}(u+v)K_2^-(v) = K_2^-(v)\mathcal{R}_{12}(u+v)K_1^-(u)\mathcal{R}_{21}(u-v) \tag{2.5}$$

remain the same as in the non-graded cases and we only need to change the usual tensor product to the graded tensor product.

In general, the dual reflection equation which depends on the unitarity and cross-unitarity relations of the $\mathcal{R}$-matrix takes different forms for different models. For the models considered in this paper, we write the graded dual reflection equation in the form introduced by Zhou *et al* [30] (see also [31]):

$$\begin{aligned}
&\mathcal{R}_{21}^{st_1st_2}(-u+v)\left(K_1^+\right)^{st_1}(u)M_1^{-1}\mathcal{R}_{12}^{st_1st_2}(-u-v-2\rho)M_1\left(K_2^+\right)^{st_2}(v)\\
&\quad = \left(K_2^+\right)^{st_2}(v)M_1\mathcal{R}_{12}^{st_1st_2}(-u-v-2\rho)M_1^{-1}\left(K_1^+\right)^{st_1}(u)\mathcal{R}_{21}^{st_1st_2}(-u+v),
\end{aligned} \tag{2.6}$$

and we will choose a common parity assignment: $p(1) = p(3) = 0$ and $p(2) = 1$, the BFB grading. Moreover, we have verified that the three different sets of equations associated with the gradings BFB, FBB and BBF are mutually equivalent and consequently, the BA results presented in this paper do not depend on our grading choice.

Now, using the relations

$$\mathcal{R}_{12}^{st_1 st_2}(u) = I_1 R_{21}(u) I_1, \qquad \mathcal{R}_{21}^{st_1 st_2}(u) = I_1 R_{12}(u) I_1 \qquad \text{and}$$
$$I K^{+}(u) I = K^{+}(u) \tag{2.7}$$

with $I = \mathrm{diag}(1, -1, 1)$ and the property $[M_1 M_2, \mathcal{R}(u)] = 0$ we can see that the usual isomorphism [32]

$$K^{-}(u) :\rightarrow K^{+}(u) = K^{-}(-u-\rho)^{st} M \tag{2.8}$$

holds with the BFB grading. Here st_i denotes super-transposition in the space i.

A quantum-integrable system is characterized by the monodromy matrix $T(u)$ satisfying the fundamental relation

$$R(u-v)\,[T(u) \otimes T(v)] = [T(v) \otimes T(u)]\,R(u-v), \tag{2.9}$$

where $R(u)$ is given by $R(u) = P\mathcal{R}(u)$.

In the framework of the QISM [20], the simplest monodromies have become known as $\mathcal{L}$ operators, the Lax operators, here defined by $\mathcal{L}_{aq}(u) = \mathcal{R}_{aq}(u)$, where the subscript a represents the auxiliary space, and q represents the quantum space. The monodromy matrix $T(u)$ is defined as the matrix product of N Lax operators on all sites of the lattice,

$$T(u) = \mathcal{L}_{aN}(u)\mathcal{L}_{aN-1}(u)\cdots\mathcal{L}_{a1}(u). \tag{2.10}$$

The main result for open boundaries integrability is: if the boundary equations are satisfied, then the Sklyanin's transfer matrix [14]

$$t(u) = \mathrm{str}_a\left(K^{+}(u)T(u)K^{-}(u)T^{-1}(-u)\right) \tag{2.11}$$

forms a commuting collection of operators in the quantum space

$$[t(u), t(v)] = 0, \qquad \forall u, v. \tag{2.12}$$

The commutativity of $t(u)$ can be proved by using the unitarity and crossing-unitarity relations, the reflection equation and the dual reflection equation. In particular, it implies the integrability of an open quantum spin chain whose Hamiltonian (with $K^{-}(0) = 1$) is given by [14]

$$H = \sum_{k=1}^{N-1} H_{k,k+1} + \frac{1}{2}\left.\frac{\mathrm{d}K_1^{-}(u)}{\mathrm{d}u}\right|_{u=0} + \frac{\mathrm{str}_0 K_0^{+}(0) H_{N,0}}{\mathrm{str}K^{+}(0)}, \tag{2.13}$$

where the two-site terms are given by

$$H_{k,k+1} = \left.\frac{\mathrm{d}}{\mathrm{d}u} P_{k,k+1}\mathcal{R}_{k,k+1}(u)\right|_{u=0} \tag{2.14}$$

in the standard fashion.

The three-state vertex models that we will consider are the $sl(2|1)^{(2)}$ model and the $osp(1|2)$ model. Their $\mathcal{R}$-matrices have a common form

$$\mathcal{R}(u) = \left(\begin{array}{ccc|ccc|ccc}
x_1 & & & & & & & & \\
 & x_2 & & x_5 & & & & & \\
 & & x_3 & & x_6 & & x_7 & & \\
\hline
 & y_5 & & x_2 & & & & & \\
 & & y_6 & & x_4 & & x_6 & & \\
 & & & & & x_2 & & x_5 & \\
\hline
 & & y_7 & & y_6 & & x_3 & & \\
 & & & & & y_5 & & x_2 & \\
 & & & & & & & & x_1
\end{array}\right) \tag{2.15}$$

satisfying the properties

$$\begin{aligned} \text{regularity}:&\quad \mathcal{R}_{12}(0) = f(0)^{1/2} P_{12} \\ \text{unitarity}:&\quad \mathcal{R}_{12}(u)\mathcal{R}_{12}^{st_1 st_2}(-u) = f(u), \\ \text{PT-symmetry}:&\quad P_{12}\mathcal{R}_{12}(u)P_{12} = \mathcal{R}_{12}^{st_1 st_2}(u), \\ \text{crossing-symmetry}:&\quad \mathcal{R}_{12}(u) = U_1 \mathcal{R}_{12}^{st_2}(-u-\rho)U_1^{-1}, \end{aligned} \tag{2.16}$$

where $f(u) = x_1(u)x_1(-u)$, $x_1(u)$ being defined for each model below. ρ is the crossing parameter and U determines the crossing matrix

$$M = U^t U = M^t. \tag{2.17}$$

Unitarity and crossing-symmetry together imply the useful relation

$$M_1 \mathcal{R}_{12}^{st_2}(-u-\rho)M_1^{-1}\mathcal{R}_{12}^{st_1}(u-\rho) = f'(u). \tag{2.18}$$

2.1. The $sl(2|1)^{(2)}$ model

The solution of the graded YB equation corresponding to $sl(2|1)^{(2)}$ in the fundamental representation has the form (2.15) with non-zero entries [33, 34]:

$$\begin{aligned} &x_1(u) = \sinh(u+2\eta)\cosh(u+\eta), && x_2(u) = \sinh u \cosh(u+\eta) \\ &x_3(u) = \sinh u \cosh(u-\eta), && x_4(u) = \sinh u \cosh(u+\eta) - \sinh 2\eta \cosh\eta, \\ &y_5(u) = x_5(u) = \sinh 2\eta \cosh(u+\eta), && y_6(u) = x_6(u) = \sinh 2\eta \sinh u, \\ &y_7(u) = x_7(u) = \sinh 2\eta \cosh\eta. && \end{aligned} \tag{2.19}$$

This $\mathcal{R}$-matrix is the simplest example of an graded $\mathcal{R}$ matrix of the twisted type. It is regular and unitary, with $f(u) = x_1(u)x_1(-u)$, P- and T-symmetric and crossing-symmetric with $M = 1$ and $\rho = \eta$. The graded version of the crossing-unitarity relation (2.18) is satisfied with $f'(u) = x_1\big(u + \mathrm{i}\frac{\pi}{2}\big)x_1\big(-u - \mathrm{i}\frac{\pi}{2}\big)$.

The most general diagonal solution for $K^-(u)$ is presented in [35] and it is given by

$$K^-(u,\beta) = \begin{pmatrix} k_{11}^-(u) & & \\ & 1 & \\ & & k_{33}^-(u) \end{pmatrix} \tag{2.20}$$

with

$$k_{11}^-(u) = -\frac{\beta\sinh u + 2\cosh u}{\beta\sinh u - 2\cosh u}, \qquad k_{33}^-(u) = \frac{\beta\cosh(u+\eta) - 2\sinh(u+\eta)}{\beta\cosh(u-\eta) + 2\sinh(u-\eta)}, \tag{2.21}$$

where β is a free parameter. Due to the automorphism (2.8) the solution for $K^+(u)$ is given by $K^-\big(-u-\rho, \frac{1}{4}\alpha\big)$ i.e.

$$K^+(u,\alpha) = \begin{pmatrix} k_{11}^+(u) & & \\ & 1 & \\ & & k_{33}^+(u) \end{pmatrix}, \tag{2.22}$$

where

$$k_{11}^+(u) = \frac{\alpha\cosh(u+\eta) - 2\sinh(u+\eta)}{\alpha\cosh(u+\eta) + 2\sinh(u+\eta)}, \qquad k_{33}^+(u) = -\frac{\alpha\sinh u + 2\cosh u}{\alpha\sinh(u+2\eta) - 2\cosh(u+2\eta)} \tag{2.23}$$

and α is another free parameter.

2.2. The $osp(2|1)$ *model*

The trigonometric solution of the graded YB equation corresponding to $osp(1|2)$ in the fundamental representation has the form (2.15) with non-zero entries [34]:

$$
\begin{aligned}
&x_1(u) = \sinh(u+2\eta)\sinh(u+3\eta), && x_2(u) = \sinh u \sinh(u+3\eta) \\
&x_3(u) = \sinh u \sinh(u+\eta), && x_4(u) = \sinh u \sinh(u+3\eta) - \sinh 2\eta \sinh 3\eta \\
&x_5(u) = \mathrm{e}^{-u}\sinh 2\eta \sinh(u+3\eta), && y_5(u) = \mathrm{e}^{u}\sinh 2\eta \sinh(u+3\eta) \\
&x_6(u) = -\mathrm{e}^{-u-2\eta}\sinh 2\eta \sinh u, && y_6(u) = \mathrm{e}^{u+2\eta}\sinh 2\eta \sinh u \\
&x_7(u) = \mathrm{e}^{-u}\sinh 2\eta(\sinh(u+3\eta) + \mathrm{e}^{-\eta}\sinh u) && \\
&y_7(u) = \mathrm{e}^{u}\sinh 2\eta\,(\sinh(u+3\eta) + \mathrm{e}^{\eta}\sinh u) &&
\end{aligned}
\tag{2.24}
$$

This $\mathcal{R}$-matrix is regular and unitary, with $f^{'}(u) = f(u) = x_1(u)x_1(-u)$. It is PT-symmetric and crossing-symmetric, with $\rho = 3\eta$ and

$$
M = \begin{pmatrix} \mathrm{e}^{-2\eta} & & \\ & 1 & \\ & & \mathrm{e}^{2\eta} \end{pmatrix}. \tag{2.25}
$$

Diagonal solutions for $K^-(u)$ have been obtained in [36]. It turns out that there are three solutions without free parameters, being $K^-(u) = 1$, $K^-(u) = F^+$ and $K^-(u) = F^-$, with

$$
F^{\pm} = \begin{pmatrix} \mp\mathrm{e}^{-2u} f^{(\pm)}(u) & & \\ & 1 & \\ & & \mp\mathrm{e}^{2u} f^{(\pm)}(u) \end{pmatrix} \tag{2.26}
$$

where we have defined

$$
f^{(+)}(u) = \frac{\sinh(u+3\eta/2)}{\sinh(u-3\eta/2)}, \qquad f^{(-)}(u) = \frac{\cosh(u+3\eta/2)}{\cosh(u-3\eta/2)}. \tag{2.27}
$$

By the automorphism (2.8), three solutions $K^+(u)$ follow as $K^+(u) = M$, $K^+(u) = G^+$ and $K^+(u) = G^-$, with

$$
G^{\pm} = \begin{pmatrix} \mp\mathrm{e}^{2u+4\eta} g^{(\pm)}(u) & & \\ & 1 & \\ & & \mp\mathrm{e}^{-2u-4\eta} g^{(\pm)}(u) \end{pmatrix}, \tag{2.28}
$$

where we have defined

$$
g^{(+)}(u) = \frac{\sinh(u+3\eta/2)}{\sinh(u+9\eta/2)}, \qquad g^{(-)}(u) = \frac{\cosh(u+3\eta/2)}{\cosh(u+9\eta/2)}. \tag{2.29}
$$

We have thus nine possibilities for the commuting transfer matrix (2.11). We will only consider three types of boundary solutions, one for each pair $(K^-(u), K^+(u))$ defined by the automorphism (2.8): $(1, M)$, (F^+, G^+) and (F^-, G^-).

3. Algebraic Bethe ansatz

In this section we have restricted ourselves in presenting only the main expressions for the open algebraic BA. More details can be obtained [26].

3.1. The reference state

Firstly, we write de double-monodromy matrix $U(u)$ as

$$U(u) = T(u)K^{-}(u)T^{-1}(-u) = \begin{pmatrix} U_{11}(u) & U_{12}(u) & U_{13}(u) \\ U_{21}(u) & U_{22}(u) & U_{23}(u) \\ U_{31}(u) & U_{32}(u) & U_{33}(u) \end{pmatrix} \tag{3.1}$$

For the vertex models considered in this paper we can choose the highest weight vector of the monodromy matrix in a lattice of N sites as the even (bosonic) completely unoccupied state

$$|0\rangle = \prod_{k=1}^{N} \overset{s}{\otimes} |0\rangle_k\,, \qquad |0\rangle_k = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \tag{3.2}$$

where $|0\rangle_k$ is the local reference state at the kth lattice site with three components. Applying $U(u)$ on $|0\rangle$ we can find

$$\begin{aligned} &U_{ii}(u)\,|0\rangle \neq 0 \qquad (i = 1, 2, 3) \\ &U_{ij}(u)\,|0\rangle = 0, \qquad (i > j), \qquad U_{ij}(u)\,|0\rangle \neq \{0, |0\rangle\}\,, \quad (i < j). \end{aligned} \tag{3.3}$$

Introducing new operators:

$$\begin{aligned} &\mathcal{D}_1(u) = U_{11}(u), \qquad \mathcal{B}_1(u) = U_{12}(u), \qquad \mathcal{B}_2(u) = U_{13}(u) \\ &\mathcal{C}_1(u) = U_{21}(u), \qquad \mathcal{D}_2(u) = U_{22}(u) - f_1(u)\mathcal{D}_1(u), \qquad \mathcal{B}_3(u) = U_{23}(u) \\ &\mathcal{C}_2(u) = U_{31}(u), \qquad \mathcal{C}_3(u) = U_{32}(u), \qquad \mathcal{D}_3(u) = U_{33}(u) - f_2(u)\mathcal{D}_1(u) - f_3(u)\mathcal{D}_2(u), \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} f_1(u) &= \frac{y_5(2u)}{x_1(2u)}, \qquad f_2(u) = \frac{y_7(2u)}{x_1(2u)}, \\ f_3(u) &= -\frac{x_1(2u)y_5(2u) - x_5(2u)y_7(2u)}{x_1(2u)x_4(2u) + x_5(2u)y_5(2u)}, \\ f_4(u) &= \frac{x_4(2u)y_7(2u) + y_5^2(2u)}{x_1(2u)x_4(2u) + x_5(2u)y_5(2u)}. \end{aligned} \tag{3.5}$$

The action of $U(u) \to \mathcal{U}(u)$ on the reference state has the usual BA form

$$\mathcal{U}(u)\,|0\rangle = \begin{pmatrix} \mathcal{X}_1(u)\,|0\rangle & * & ** \\ 0 & \mathcal{X}_2(u)\,|0\rangle & *** \\ 0 & 0 & \mathcal{X}_3(u)\,|0\rangle \end{pmatrix}, \tag{3.6}$$

where

$$\begin{aligned} \mathcal{X}_1(u) &= k_{11}^{-}(u)\frac{x_1^{2N}(u)}{f^N(u)}, \\ \mathcal{X}_2(u) &= [k_{22}^{-}(u) - k_{11}^{-}(u)f_1(u)]\frac{x_2^{2N}(u)}{f^N(u)}, \\ \mathcal{X}_3(u) &= [k_{33}^{-}(u) - k_{22}^{-}(u)f_3(u) - k_{11}^{-}(u)f_4(u)]\frac{x_3^{2N}(u)}{f^N(u)}. \end{aligned} \tag{3.7}$$

The transfer matrix $t(u) = \mathrm{str}(K^{+}U)$, with diagonal left reflection $K^{(+)} = \mathrm{diag}(k_{11}^{+}, k_{22}^{+}.k_{33}^{+})$ and BFB grading, has the form

$$\begin{aligned} t(u) &= k_{11}^{+}(u)U_{11}(u) - k_{22}^{+}(u)U_{22}(u) + k_{33}^{+}(u)U_{33}(u) \\ &= \Omega_1(u)\mathcal{D}_1(u) + \Omega_2(u)\mathcal{D}_2(u) + \Omega_3(u)\mathcal{D}_3(u), \end{aligned} \tag{3.8}$$

where

$$\begin{aligned}
\Omega_1(u) &= k_{11}^{+}(u) - f_1(u)k_{22}^{+}(u) + f_2(u)k_{33}^{+}(u), \\
\Omega_2(u) &= -k_{22}^{+}(u) + f_3(u)k_{33}^{+}(u), \\
\Omega_3(u) &= k_{33}^{+}(u).
\end{aligned} \tag{3.9}$$

3.2. The fundamental commutation relations

Taking into account the fundamental relation

$$\mathcal{R}_{12}(u-v)\mathcal{U}_1(u)\mathcal{R}_{21}(u+v)\mathcal{U}_2(v) = \mathcal{U}_2(v)\mathcal{R}_{12}(u+v)\mathcal{U}_1(u)\mathcal{R}_{21}(u-v), \tag{3.10}$$

we can obtain the following commutation relations necessary to fix the one-particle state $\Psi_1(u_1) = \mathcal{B}_1(u_1)\,|0\rangle$ as an eigenstate of the transfer matrix $t(u)$ (3.8):

$$\begin{aligned}
\mathcal{D}_1(u)\mathcal{B}_1(u_1) &= a_{11}(u,u_1)\mathcal{B}_1(u_1)\mathcal{D}_1(u) + a_{12}(u,u_1)\mathcal{B}_1(u)\mathcal{D}_1(u_1) + a_{13}(u,u_1)\mathcal{B}_1(u)\mathcal{D}_2(u_1) \\
&\quad + a_{14}(u,u_1)\mathcal{B}_2(u)\mathcal{C}_1(u_1) + a_{15}(u,u_1)\mathcal{B}_2(u)\mathcal{C}_3(u_1) + a_{16}(u,u_1)\mathcal{B}_2(u_1)\mathcal{C}_1(u),
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
\mathcal{D}_2(u)\mathcal{B}_1(u_1) &= a_{21}(u,u_1)\mathcal{B}_1(u_1)\mathcal{D}_2(u) + a_{22}(u,u_1)\mathcal{B}_1(u)\mathcal{D}_1(u_1) + a_{23}(u,u_1)\mathcal{B}_1(u)D_2(u_l) \\
&\quad + a_{24}(u,u_1)\mathcal{B}_3(u)\mathcal{D}_1(u_1) + a_{25}(u,u_1)\mathcal{B}_3(u)\mathcal{D}_2(u_1) + a_{26}(u,u_1)\mathcal{B}_2(u)\mathcal{C}_1(u_1) \\
&\quad + a_{27}(u,u_1)\mathcal{B}_2(u)\mathcal{C}_3(u_1) + a_{28}(u,u_1)\mathcal{B}_2(u_1)\mathcal{C}_1(u) + a_{29}(u,u_1)\mathcal{B}_2(u_1)\mathcal{C}_3(u),
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
\mathcal{D}_3(u)\mathcal{B}_1(u_1) &= a_{31}(u,u_1)\mathcal{B}_1(u_1)\mathcal{D}_3(u) + a_{32}(u,u_1)\mathcal{B}_1(u)\mathcal{D}_1(u_1) + a_{33}(u,u_1)\mathcal{B}_1(u)\mathcal{D}_2(u_1) \\
&\quad + a_{34}(u,u_1)\mathcal{B}_3(u)\mathcal{D}_1(u_1) + a_{35}(u,u_1)\mathcal{B}_3(u)\mathcal{D}_2(u_1) + a_{36}(u,u_1)\mathcal{B}_2(u)\mathcal{C}_1(u_1) \\
&\quad + a_{37}(u,u_1)\mathcal{B}_2(u)\mathcal{C}_3(u_1) + a_{38}(u,u_1)\mathcal{B}_2(u_1)\mathcal{C}_1(u) + a_{39}(u,u_1)\mathcal{B}_2(u_1)\mathcal{C}_3(u).
\end{aligned} \tag{3.13}$$

For the two-particle eigenstate of $t(u)$

$$\begin{aligned}
\Psi_2(u_1,u_2) &= \mathcal{B}_1(u_1)\mathcal{B}_1(u_2)\,|0\rangle + G_{d_1}(u_1,u_2)\mathcal{B}_2(u_1)\mathcal{D}_1(u_2)\,|0\rangle \\
&\quad + G_{d_2}(u_1,u_2)\mathcal{B}_2(u_1)\mathcal{D}_2(u_2)\,|0\rangle,
\end{aligned} \tag{3.14}$$

the necessary commutation relations are

$$\begin{aligned}
\mathcal{B}_1(u_1)\mathcal{B}_1(u_2) &= \omega(u_1,u_2)[\mathcal{B}_1(u_2)\mathcal{B}_1(u_1) + G_{d_1}(u_2,u_1)\mathcal{B}_2(u_2)\mathcal{D}_1(u_1) \\
&\quad + G_{d_2}(u_2,u_1)\mathcal{B}_2(u_2)\mathcal{D}_2(u_1)] - G_{d_1}(u_1,u_2)\mathcal{B}_2(u_1)\mathcal{D}_1(u_2) \\
&\quad - G_{d_2}(u_1,u_2)\mathcal{B}_2(u_1)\mathcal{D}_2(u_2),
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
\mathcal{D}_1(u)\mathcal{B}_2(v) &= b_{11}(u,v)\mathcal{B}_2(v)\mathcal{D}_1(u) + b_{12}(u,v)\mathcal{B}_2(u)\mathcal{D}_1(v) + b_{13}(u,v)\mathcal{B}_2(u)\mathcal{D}_2(v) \\
&\quad + b_{14}(u,v)\mathcal{B}_2(u)\mathcal{D}_3(v) + b_{15}(u,v)\mathcal{B}_1(u)\mathcal{B}_1(v) + b_{16}(u,v)\mathcal{B}_1(u)\mathcal{B}_3(v),
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
\mathcal{D}_2(u)\mathcal{B}_2(v) &= b_{21}(u,v)\mathcal{B}_2(v)\mathcal{D}_2(u) + b_{22}(u,v)\mathcal{B}_2(u)\mathcal{D}_1(v) + b_{23}(u,v)\mathcal{B}_2(u)\mathcal{D}_2(v) \\
&\quad + b_{24}(u,v)\mathcal{B}_2(u)\mathcal{D}_3(v) + b_{25}(u,v)\mathcal{B}_1(u)\mathcal{B}_1(v) + b_{26}(u,v)\mathcal{B}_1(u)\mathcal{B}_3(v) \\
&\quad + b_{27}(u,v)\mathcal{B}_3(u)\mathcal{B}_1(v) + b_{28}(u,v)\mathcal{B}_3(u)\mathcal{B}_3(v),
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
\mathcal{D}_3(u)\mathcal{B}_2(v) &= b_{31}(u,v)\mathcal{B}_2(v)\mathcal{D}_3(u) + b_{32}(u,v)\mathcal{B}_2(u)\mathcal{D}_1(v) + b_{33}(u,v)\mathcal{B}_2(u)\mathcal{D}_2(v) \\
&\quad + b_{34}(u,v)\mathcal{B}_2(u)\mathcal{D}_3(v) + b_{35}(u,v)\mathcal{B}_1(u)\mathcal{B}_1(v) + b_{36}(u,v)\mathcal{B}_1(u)\mathcal{B}_3(v) \\
&\quad + b_{37}(u,v)\mathcal{B}_3(u)\mathcal{B}_1(v) + b_{38}(u,v)\mathcal{B}_3(u)\mathcal{B}_3(v),
\end{aligned} \tag{3.18}$$

$$\begin{aligned}\mathcal{C}_1(u)\mathcal{B}_1(v) = {}& c_{11}(u,v)\mathcal{B}_1(v)\mathcal{C}_1(u) + c_{12}(u,v)\mathcal{B}_1(v)\mathcal{C}_3(v) + c_{13}(u,v)\mathcal{B}_1(u)\mathcal{C}_3(v)\\ &+ c_{14}(u,v)\mathcal{B}_2(u)\mathcal{C}_3(v) + c_{15}(u,v)\mathcal{B}_2(v)\mathcal{C}_2(u) + c_{16}(u,v)\mathcal{D}_1(v)\mathcal{D}_1(u)\\ &+ c_{17}(u,v)\mathcal{D}_1(v)\mathcal{D}_2(u) + c_{18}(u,v)\mathcal{D}_1(u)\mathcal{D}_1(v) + c_{19}(u,v)\mathcal{D}_1(u)\mathcal{D}_2(v)\\ &+ c_{110}(u,v)\mathcal{D}_2(u)\mathcal{D}_1(v) + c_{111}(u,v)\mathcal{D}_2(u)\mathcal{D}_2(v),\end{aligned} \tag{3.19}$$

$$\begin{aligned}\mathcal{C}_3(u)\mathcal{B}_1(v) = {}& c_{21}(u,v)\mathcal{B}_1(v)\mathcal{C}_1(u) + c_{22}(u,v)\mathcal{B}_1(v)\mathcal{C}_3(v) + c_{23}(u,v)\mathcal{B}_1(u)\mathcal{C}_3(v)\\ &+ c_{24}(u,v)\mathcal{B}_2(u)\mathcal{C}_3(v) + c_{25}(u,v)\mathcal{B}_2(v)\mathcal{C}_2(u) + c_{26}(u,v)\mathcal{D}_1(v)\mathcal{D}_1(u)\\ &+ c_{27}(u,v)\mathcal{D}_1(v)\mathcal{D}_2(u) + c_{28}(u,v)\mathcal{D}_1(u)\mathcal{D}_3(v) + c_{29}(u,v)\mathcal{D}_1(u)\mathcal{D}_2(v)\\ &+ c_{210}(u,v)\mathcal{D}_1(u)\mathcal{D}_2(v) + c_{211}(u,v)\mathcal{D}_2(u)\mathcal{D}_1(v) + c_{212}(u,v)\mathcal{D}_2(u)\mathcal{D}_2(v)\\ &+ c_{213}(u,v)\mathcal{D}_3(u)\mathcal{D}_1(v) + c_{214}(u,v)\mathcal{D}_3(u)\mathcal{D}_1(v))\end{aligned} \tag{3.20}$$

where

$$\omega(u_1,u_2) = -\frac{x_3(u_1-u_2)x_4(u_1-u_2) - x_6(u_1-u_2)y_6(u_1-u_2)}{x_1(u_1-u_2)x_3(u_1-u_2)} \tag{3.21}$$

$$G_{d_1}(u_1,u_2) = \frac{x_6(u_1-u_2)}{x_3(u_1-u_2)}\frac{x_2(2u_2)}{x_1(2u_2)}, \tag{3.22}$$

$$G_{d_2}(u_1,u_2) = -\frac{x_6(u_1+u_2)}{x_2(u_1+u_2)}. \tag{3.23}$$

3.3. The n-particle state

For the general case we can seek for operator valued functions with a recurrence relation of the form

$$\begin{aligned}\Phi_n(u,\ldots,u_n) = {}& \mathcal{B}_1(u_1)\Phi_{n-1}(u_2,\ldots,u_n)\\ &+ \mathcal{B}_2(u_1)\sum_{i=2}^{n} F_1^{(i)}(u_1,\ldots,u_n)\Phi_{n-2}(u_2,\ldots,\check{u}_i,\ldots,u_n)\mathcal{D}_1(u_i)\\ &+ \mathcal{B}_2(u_1)\sum_{i=2}^{n} F_2^{(i)}(u_1,\ldots,u_n)\Phi_{n-2}(u_2,\ldots,\check{u}_i,\ldots,u_n)\mathcal{D}_2(u_i)\end{aligned} \tag{3.24}$$

with $\Phi_0 = 1$. It was shown in [25] that the above operator is normal ordered satisfying $n-1$ exchange conditions

$$\Phi_n(u_1,\ldots,u_i,u_{i+1},\ldots,u_n) = \omega(u_i,u_{i+1})\Phi_n(u_1,\ldots,u_{i+1},u_i,\ldots,u_n) \tag{3.25}$$

provided the functions $F_\alpha^{(i)}(u_1,\ldots,u_n)$ are given by

$$F_\alpha^{(i)}(u_1,\ldots,u_n) = \prod_{j=2}^{i-1}\omega(u_j,u_i)\prod_{k=2,k\neq i}^{n} a_{\alpha 1}(u_i,u_k)G_{d_\alpha}(u_1,u_i), \qquad (\alpha = 1,2). \tag{3.26}$$

Therefore the n-particle state will be given by

$$\Psi_n(u_1, \ldots, u_n) = \Phi_n(u, \ldots, u_n)\,|0\rangle \tag{3.27}$$

and the action of the operators $\mathcal{D}_\alpha(u)$, $\alpha = 1, 2, 3$, on this state will be represented by

$$\begin{aligned}
\mathcal{D}_\alpha(u)\Psi_n(u_1, \ldots, u_n) &= \mathcal{X}_\alpha(u)\prod_{i=1}^{n} a_{\alpha 1}(u, u_i)\Psi_n(u_1, \ldots, u_n) \\
&+ \sum_{i=1}^{n}\prod_{j=1}^{i-1}\omega(u_j, u_i)[\mathcal{X}_1(u)a_{\alpha 2}(u, u_i)\prod_{j\neq i}^{n} a_{11}(u, u_j) + \mathcal{X}_2(u)a_{\alpha 3}(u, u_i) \\
&\times \prod_{j\neq i}^{n} a_{21}(u, u_j)]\mathcal{B}_1(u)\Psi_{n-1}(\overset{\vee}{u}_i) + (1-\delta_{\alpha,1})\sum_{i=1}^{n}\prod_{j=1}^{i-1}\omega(u_j, u_i)[\mathcal{X}_1(u)a_{\alpha 4}(u, u_i) \\
&\times \prod_{j\neq i}^{n} a_{11}(u, u_j) + \mathcal{X}_2(u)a_{\alpha 5}(u, u_i)\prod_{j\neq i}^{n} a_{21}(u, u_j)]\mathcal{B}_3(u)\Psi_{n-1}(\overset{\vee}{u}_i) \\
&+ \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left\{\mathcal{X}_1(u_i)\mathcal{X}_1(u_j)\prod_{k\neq i,j}^{n} a_{11}(u_i, u_k)\prod_{l\neq i,j}^{n} a_{11}(u_j, u_l)H_{\alpha 1}(u_i, u_j)\right. \\
&+ \mathcal{X}_2(u_i)\mathcal{X}_1(u_j)\prod_{k\neq i,j}^{n} a_{21}(u_i, u_k)\prod_{l\neq i,j}^{n} a_{11}(u_j, u_l)H_{\alpha 2}(u_i, u_j) \\
&+ \mathcal{X}_1(u_i)\mathcal{X}_2(u_j)\prod_{k\neq i,j}^{n} a_{11}(u_i, u_k)\prod_{l\neq i,j}^{n} a_{21}(u_j, u_l)H_{\alpha 3}(u_i, u_j) \\
&\left.+ \mathcal{X}_2(u_i)\mathcal{X}_2(u_j)\prod_{k\neq i,j}^{n} a_{21}(u_i, u_k)\prod_{l\neq i,j}^{n} a_{21}(u_j, u_l)H_{\alpha 4}(u_i, u_j)\right\} \\
&\times \prod_{k=1}^{i-1}\omega(u_k, u_i)\prod_{l=1\neq i}^{i-1}\omega(u_l, u_j)\mathcal{B}_2(u)\Psi_{n-2}(\overset{\vee}{u}_i, \overset{\vee}{u}_j),
\end{aligned} \tag{3.28}$$

where

$$\begin{aligned}
H_{11}(u_1, u_2) &= a_{14}(u, u_1)\,(c_{16}(u_1, u_2) + c_{18}(u_1, u_2)) + a_{15}(u, u_1)\,(c_{26}(u_1, u_2) + c_{29}(u_1, u_2)) \\
&+ b_{12}(u, u_1)G_{d_1}(u_1, u_2) + \omega(u_1, u)a_{11}(u, u_1)a_{12}(u, u_2)G_{d_1}(u, u_1) \\
H_{12}(u_1, u_2) &= a_{14}(u, u_1)\,(c_{17}(u_1, u_2) + c_{110}(u_1, u_2)) + a_{15}(u, u_1)\,(c_{27}(u_1, u_2) + c_{211}(u_1, u_2)) \\
&+ b_{13}(u, u_1)G_{d_1}(u_1, u_2) + \omega(u_1, u)a_{11}(u, u_1)a_{12}(u, u_2)G_{d_2}(u, u_1) \\
H_{13}(u_1, u_2) &= a_{14}(u, u_1)c_{19}(u_1, u_2) + a_{15}(u, u_1)c_{210}(u_1, u_2) + b_{12}(u, u_1)G_{d_2}(u_1, u_2) \\
&+ \omega(u_1, u)a_{11}(u, u_1)a_{13}(u, u_2)G_{d_1}(u, u_1) \\
H_{14}(u_1, u_2) &= a_{14}(u, u_1)c_{111}(u_1, u_2) + a_{15}(u, u_1)c_{212}(u_1, u_2) + b_{13}(u, u_1)G_{d_2}(u_1, u_2) \\
&+ \omega(u_1, u)a_{11}(u, u_1)a_{13}(u, u_2)G_{d_2}(u, u_1)
\end{aligned} \tag{3.29}$$

and

$$\begin{aligned}
H_{j1}(u_1, u_2) &= a_{j6}(u, u_1)\,(c_{16}(u_1, u_2) + c_{18}(u_1, u_2)) + a_{j7}(u, u_1)\,(c_{26}(u_1, u_2) + c_{29}(u_1, u_2)) \\
&+ b_{j2}(u, u_1)G_{d_1}(u_1, u_2) + \omega(u_1, u)a_{j1}(u, u_1)a_{j2}(u, u_2)G_{d_1}(u, u_1) \\
&+ a_{j1}(u, u_1)a_{j4}(u, u_2)d_{13}(u_1, u) \\
H_{j2}(u_1, u_2) &= a_{j6}(u, u_1)\,(c_{17}(u_1, u_2) + c_{110}(u_1, u_2)) + a_{j7}(u, u_1)\,(c_{27}(u_1, u_2) + c_{211}(u_1, u_2))
\end{aligned}$$

$$
\begin{aligned}
&\quad + b_{j3}(u,u_1)G_{d_1}(u_1,u_2) + \omega(u_1,u)a_{j1}(u,u_1)a_{j2}(u,u_2)G_{d_2}(u,u_1) \\
&\quad + a_{j1}(u,u_1)a_{j4}(u,u_2)d_{14}(u_1,u) \\
H_{j3}(u_1,u_2) &= a_{j6}(u,u_1)c_{19}(u_1,u_2) + a_{j7}(u,u_1)c_{210}(u_1,u_2) + b_{j2}(u,u_1)G_{d_2}(u_1,u_2) \\
&\quad + \omega(u_1,u)a_{j1}(u,u_1)a_{j3}(u,u_2)G_{d_1}(u,u_1) + a_{j1}(u,u_1)a_{j5}(u,u_2)d_{13}(u_1,u) \\
H_{j4}(u_1,u_2) &= a_{j6}(u,u_1)c_{111}(u_1,u_2) + a_{j7}(u,u_1)c_{212}(u_1,u_2) + b_{j3}(u,u_1)G_{d_2}(u_1,u_2) \\
&\quad + \omega(u_1,u)a_{j1}(u,u_1)a_{j3}(u,u_2)G_{d_2}(u,u_1) + a_{j1}(u,u_1)a_{j5}(u,u_2)d_{14}(u_1,u).
\end{aligned} \tag{3.30}
$$

Finally, the corresponding n-particle eigenvalue problem will be

$$
t(u)\Psi_n(u_1,\ldots,u_n) = \left(\sum_{\alpha=1}^{3} \Omega_\alpha(u)\mathcal{X}_\alpha(u)\prod_{i=1}^{n} a_{\alpha 1}(u,u_i)\right)\Psi_n(u_1,\ldots,u_n) \tag{3.31}
$$

provided that the BA equations are satisfied

$$
\frac{\mathcal{X}_1(u_k)}{\mathcal{X}_2(u_k)} = \Theta(u_k)\prod_{j=1, j\neq k}^{n} \frac{a_{21}(u_k,u_j)}{a_{11}(u_k,u_j)}, \qquad (k=1,2,\ldots,n), \tag{3.32}
$$

where

$$
\Theta(u_k) = -\frac{\Omega_2(u)a_{25}(u,u_k) + \Omega_3(u)a_{35}(u,u_k)}{\Omega_2(u)a_{24}(u,u_k) + \Omega_3(u)a_{34}(u,u_k)}. \tag{3.33}
$$

4. Explicit solutions

In this section explicit expressions of the eigenvalue problem are presented for both models,. First we recall the fundamental relation (3.10) to get the coefficients $a_{ij}(u,v)$ which appear effectively in the BA expressions (3.31) and (3.32):

$$
\begin{aligned}
a_{11}(u,v) &= \frac{x_1(u-v)}{x_2(u-v)}\frac{x_2(u+v)}{x_1(u+v)} \\
a_{21}(u,v) &= -\omega(u,v)\left[\frac{x_1(u+v)x_4(u+v) + x_5(u+v)y_5(u+v)}{x_1(u+v)x_2(u+v)}\right] \\
a_{31}(u,v) &= \frac{x_2(u-v)}{x_3(u-v)}\left[\frac{x_2(u+v)^2 + x_6(u+v)y_6(u+v)}{x_2(u+v)x_3(u+v)}\right]
\end{aligned} \tag{4.1}
$$

$$
\begin{aligned}
a_{24}(u,v) &= -\left[\frac{x_6(u-v)}{x3(u-v)}\frac{x_3(u+v)}{x_2(u+v)} - f_1(v)\frac{x_6(u+v)}{x_2(u+v)}\right] \\
a_{25}(u,v) &= \frac{x_6(u+v)}{x_2(u+v)} \\
a_{34}(u,v) &= -f_3(u)\left[f_1(v)\frac{x_6(u+v)}{x_2(u+v)} - \frac{x_6(u-v)}{x_3(u-v)}\frac{x_3(u+v)}{x_2(u+v)}\right] \\
&\quad + f_1(v)\frac{y_6(u-v)}{x_3(u-v)}\left[\frac{x_6(u+v)y_6(u+v) + x_2^2(u+v)}{x_2(u+v)x_3(u+v)}\right] \\
&\quad - \left[\frac{x_6(u-v)y_6(u-v) + x_2^2(u-v)}{x_3^2(u-v)}\right]\frac{y_6(u+v)}{x_2(u+v)} \\
a_{35}(u,v) &= -f_3(u)\frac{x_6(u+v)}{x_2(u+v)} + \frac{y_6(u-v)}{x_3(u-v)}\left[\frac{x_6(u+v)y_6(u+v) + x_2^2(u+v)}{x_2(u+v)x_3(u+v)}\right]
\end{aligned} \tag{4.2}
$$

with the $f_i(u)$ given by (3.5).

4.1. $sl(2|1)^{(2)}$ model

Substituting the matrix elements of the $\mathcal{R}$ matrix (2.19) and the matrix elements of the K matrices (2.21) and (2.23) for this model we get the following expressions:

$$\Omega_1(u) = \frac{\cosh(2u+3\eta)}{\cosh(2u+\eta)} \frac{\alpha\sinh u - 2\cosh u}{\alpha\sinh(u+2\eta) - 2\cosh(u+2\eta)} \frac{\alpha\cosh(u+\eta) - 2\sinh(u+\eta)}{\alpha\cosh(u+\eta) + 2\sinh(u+\eta)}, \tag{4.3}$$

$$\Omega_2(u) = -\frac{\sinh(2u+2\eta)}{\sinh(2u)} \frac{\alpha\sinh u - 2\cosh u}{\alpha\sinh(u+2\eta) - 2\cosh(u+2\eta)}, \tag{4.4}$$

$$\Omega_3(u) = -\frac{\alpha\sinh u + 2\cosh u}{\alpha\sinh(u+2\eta) - 2\cosh(u+2\eta)}, \tag{4.5}$$

$$\mathcal{X}_1(u) = -\frac{\beta\sinh u + 2\cosh u}{\beta\sinh u - 2\cosh u} \frac{x_1^{2N}(u)}{f^N(u)}, \tag{4.6}$$

$$\mathcal{X}_2(u) = \frac{\sinh(2u)}{\sinh(2u+2\eta)} \frac{\beta\sinh(u+2\eta) - 2\cosh(u+2\eta)}{\beta\sinh u - 2\cosh u} \frac{x_2^{2N}(u)}{f^N(u)}, \tag{4.7}$$

$$\begin{aligned}\mathcal{X}_3(u) = {} & \frac{\cosh(2u-\eta)}{\cosh(2u+\eta)} \frac{\beta\sinh(u+2\eta) - 2\cosh(u+2\eta)}{\beta\sinh u - 2\cosh u} \\ & \times \frac{\beta\cosh(u+\eta) - 2\sinh(u+\eta)}{\beta\cosh(u-\eta) + 2\sinh(u-\eta)} \frac{x_3^{2N}(u)}{f^N(u)},\end{aligned} \tag{4.8}$$

and

$$\Theta(u_i) = \frac{\sinh(2u_i+2\eta)}{\sinh(2u_i)} \frac{\alpha\cosh(u_i+\eta) + 2\sinh(u_i+\eta)}{\alpha\cosh(u_i+\eta) - 2\sinh(u_i+\eta)}. \tag{4.9}$$

From these data one can see that the n-particle state $\Psi_n(\{u_i\})$ is an eigenfuction of the transfer matrix (3.8) for the $sl(2|1)^{(2)}$ vertex model with eigenvalue

$$\begin{aligned}\Lambda_n(u,\{u_i\}) = {} & \Omega_1(u)\mathcal{X}_1(u) \prod_{i=1}^{n} \frac{\sinh(u+u_i-\eta)}{\sinh(u+u_i+\eta)} \frac{\sinh(u-u_i-\eta)}{\sinh(u-u_i+\eta)} \\ & - \Omega_2(u)\mathcal{X}_2(u) \prod_{i=1}^{n} \frac{\sinh(u+u_i-\eta)}{\sinh(u+u_i+\eta)} \frac{\sinh(u-u_i-\eta)}{\sinh(u-u_i+\eta)} \\ & \times \frac{\cosh(u+u_i+2\eta)}{\cosh(u+u_i)} \frac{\cosh(u-u_i+2\eta)}{\cosh(u-u_i)} \\ & + \Omega_3(u)\mathcal{X}_3(u) \prod_{i=1}^{n} \frac{\cosh(u+u_i+2\eta)}{\cosh(u+u_i)} \frac{\cosh(u-u_i+2\eta)}{\cosh(u-u_i)}\end{aligned} \tag{4.10}$$

provided that the parameters $\{u_i\}$ satisfy the BA equations

$$\begin{aligned}\left(\frac{\sinh(u_i+\eta}{\sinh(u_i-\eta)}\right)^{2N} = {} & -\frac{\alpha\cosh u_i + 2\sinh u_i}{\alpha\cosh u_i - 2\sinh u_i} \frac{\beta\sinh(u_i+\eta) - 2\cosh(u_i+\eta)}{\beta\sinh(u_i-\eta) + 2\cosh(u_i-\eta)} \\ & \times \prod_{\{j\neq i\}=1}^{n} \frac{\cosh(u_i+u_j+\eta)}{\cosh(u_i+u_j-\eta)} \frac{\cosh(u_i-u_j+\eta)}{\cosh(u_i-u_j-\eta)} \qquad i = 1, 2, \ldots, n,\end{aligned} \tag{4.11}$$

where we have used the shifts $u_i \to u_i = u_i - \eta$ to bring these expressions into a symmetric form in η.

The formulation of this model in terms of the QISM presented here is new. However, one can verify that our results give the energy eigenspectrum previously obtained in the framework of coordinate BA by Fireman *et al* [29].

4.2. osp(2|1) *model*

For this model the K matrices have no free parameters but we have to consider three cases.

4.2.1. The (1, M) case. In this case we have

$$\begin{aligned}
\Omega_1(u) &= \frac{\sinh(2u+\eta)}{\sinh(2u+2\eta)}\frac{\sinh(2u+6\eta)}{\sinh(2u+3\eta)} \\
\Omega_2(u) &= -\mathrm{e}^{2\eta}\frac{\sinh(2u+6\eta)}{\sinh(2u+4\eta)} \\
\Omega_3(u) &= \mathrm{e}^{2\eta}
\end{aligned} \tag{4.12}$$

$$\begin{aligned}
\mathcal{X}_1(u) &= \frac{x_1^{2N}(u)}{f^N(u)} \\
\mathcal{X}_2(u) &= \mathrm{e}^{-2\eta}\frac{\sinh(2u)}{\sinh(2u+2\eta)}\frac{x_2^{2N}(u)}{f^N(u)} \\
\mathcal{X}_3(u) &= \mathrm{e}^{-2\eta}\frac{\sinh(2u)}{\sinh(2u+4\eta)}\frac{\sinh(2u+5\eta)}{\sinh(2u+3\eta)}\frac{x_3^{2N}(u)}{f^N(u)}
\end{aligned} \tag{4.13}$$

and

$$\Theta(u_i) = \mathrm{e}^{2\eta}\frac{\sinh(2u_i+2\eta)}{\sinh(2u_i)}, \qquad i = 1,\ldots,n. \tag{4.14}$$

Therefore, the n-particle state $\Psi_n(\{u_i\})$ is an eigenfuction of the transfer matrix (3.8) for the $osp(2|1)$ vertex model with boundaries $(1, M)$. The corresponding eigenvalue is given by

$$\begin{aligned}
\Lambda_n(u,\{u_i\}) = {}& \frac{\sinh(2u+\eta)}{\sinh(2u+2\eta)}\frac{\sinh(2u+6\eta)}{\sinh(2u+3\eta)}\frac{x_1^{2N}(u)}{f^N(u)}\prod_{i=1}^{n}\frac{\sin(u+u_i)}{\sinh(u+u_i+2\eta)}\frac{\sinh(u-u_i-2\eta)}{\sinh(u-u_i)} \\
& - \frac{\sinh(2u+6\eta)}{\sinh(2u+4\eta)}\frac{\sinh(2u)}{\sinh(2u+2\eta)}\frac{x_2^{2N}(u)}{f^N(u)}\prod_{i=1}^{n}\frac{\sin(u+u_i+4\eta)}{\sinh(u+u_i+3\eta)}\frac{\sinh(u+u_i+\eta)}{\sinh(u+u_i+2\eta)} \\
& \times \frac{\sin(u-u_i+2\eta)}{\sinh(u-u_i)}\frac{\sinh(u-u_i-\eta)}{\sinh(u-u_i+\eta)} \\
& + \frac{\sinh(2u)}{\sinh(2u+4\eta)}\frac{\sinh(2u+5\eta)}{\sinh(2u+3\eta)}\frac{x_3^{2N}(u)}{f^N(u)}\prod_{i=1}^{n}\frac{\sinh(u+u_i+5\eta)}{\sinh(u+u_i+3\eta)}\frac{\sinh(u-u_i+3\eta)}{\sinh(u-u_i+\eta)}
\end{aligned} \tag{4.15}$$

provided that its parameters $\{u_i\}$ are solutions of the BA equations

$$\begin{aligned}
\left(\frac{\sinh(u_i+2\eta)}{\sinh u_i}\right)^{2N} = {}& \prod_{\{j\neq i\}=1}^{n}\frac{\sin(u_i+u_j+4\eta)}{\sinh(u_i+u_j+3\eta)}\frac{\sinh(u_i+u_j+\eta)}{\sin(u_i+u_j)}\frac{\sin(u_i-u_j+2\eta)}{\sinh(u_i-u_j-2\eta)} \\
& \times \frac{\sinh(u_i-u_j-\eta)}{\sinh(u_i-u_j+\eta)}, \qquad i = 1,\ldots,n
\end{aligned} \tag{4.16}$$

4.2.2. The (F^+, G^+) *case.* Here we have

$$\Omega_1(u) = -\mathrm{e}^{2u}\frac{\sinh(2u+6\eta)}{\sinh(2u+2\eta)}\frac{\sinh\left(u+\frac{5}{2}\eta\right)}{\sinh\left(u+\frac{9}{2}\eta\right)}\frac{\cosh\left(u+\frac{1}{2}\eta\right)}{\cosh\left(u+\frac{3}{2}\eta\right)}$$

$$\Omega_2(u) = -\frac{\sinh(2u+6\eta)}{\sinh(2u+4\eta)}\frac{\sinh\left(u+\frac{5}{2}\eta\right)}{\sinh\left(u+\frac{9}{2}\eta\right)}$$
$$\Omega_3(u) = -\mathrm{e}^{-2u-4\eta}\frac{\sinh\left(u+\frac{3}{2}\eta\right)}{\sinh\left(u+\frac{9}{2}\eta\right)} \tag{4.17}$$

$$\mathcal{X}_1(u) = -\mathrm{e}^{-2u}\frac{\sinh\left(u+\frac{3}{2}\eta\right)}{\sinh\left(u-\frac{3}{2}\eta\right)}\frac{x_1^{2N}(u)}{f^N(u)}$$
$$\mathcal{X}_2(u) = \frac{\sinh(2u)}{\sinh(2u+2\eta)}\frac{\sinh\left(u+\frac{1}{2}\eta\right)}{\sinh\left(u-\frac{3}{2}\eta\right)}\frac{x_2^{2N}(u)}{f^N(u)}$$
$$\mathcal{X}_3(u) = -\mathrm{e}^{2u+4\eta}\frac{\sinh(2u)}{\sinh(2u+4\eta)}\frac{\sinh\left(u+\frac{1}{2}\eta\right)}{\sinh\left(u-\frac{3}{2}\eta\right)}\frac{\cosh\left(u+\frac{5}{2}\eta\right)}{\cosh\left(u+\frac{3}{2}\eta\right)}\frac{x_3^{2N}(u)}{f^N(u)} \tag{4.18}$$

and

$$\Theta(u_i) = -\mathrm{e}^{-2u_i}\frac{\sinh(2u_i+2\eta)}{\sinh(2u_i)}\frac{\sinh\left(u_i+\frac{1}{2}\eta\right)}{\sinh\left(u_i+\frac{3}{2}\eta\right)} \tag{4.19}$$

Therefore, the n-particle state $\Psi_n(\{u_i\})$ is an eigenfuction of the transfer matrix (3.8) for the $osp(2|1)$ vertex model with boundaries (F^+, G^+). The corresponding eigenvalue is given by

$$\begin{aligned}\Lambda_n(u,\{u_i\}) = {} & \Omega_1(u)\mathcal{X}_1(u)\prod_{i=1}^{n}\frac{\sin(u+u_i-\eta)}{\sinh(u+u_i+\eta)}\frac{\sinh(u-u_i-\eta)}{\sinh(u-u_i+\eta)}\\ & +\Omega_2(u)\mathcal{X}_2(u)\prod_{i=1}^{n}\frac{\sin(u+u_i+3\eta)}{\sinh(u+u_i+2\eta)}\frac{\sin(u-u_i+3\eta)}{\sinh(u-u_i+\eta)}\\ & \times\frac{\sinh(u+u_i)}{\sinh(u+u_i+\eta)}\frac{\sinh(u-u_i)}{\sinh(u-u_i+2\eta)}\\ & +\Omega_3(u)\mathcal{X}_3(u)\prod_{i=1}^{n}\frac{\sinh(u+u_i+4\eta)}{\sinh(u+u_i+2\eta)}\frac{\sinh(u-u_i+4\eta)}{\sinh(u-u_i+2\eta)}\end{aligned} \tag{4.20}$$

with the BA equations

$$\begin{aligned}\left(\frac{\sinh(u_i+\eta)}{\sinh(u_i-\eta)}\right)^{2N} = {} & \left(\frac{\sinh\left(u_i-\frac{1}{2}\eta\right)}{\sinh\left(u_i+\frac{1}{2}\eta\right)}\right)^2\prod_{\{j\neq i\}=1}^{n}\frac{\sin(u_i+u_j+2\eta)}{\sinh(u_i+u_j+\eta)}\frac{\sin(u_i-u_j+2\eta)}{\sinh(u_i-u_j-2\eta)}\\ & \times\frac{\sinh(u_i+u_j-\eta)}{\sin(u_i+u_j+\eta)}\frac{\sinh(u_i-u_j-\eta)}{\sinh(u_i-u_j+\eta)}, \qquad i=1,\ldots,n.\end{aligned} \tag{4.21}$$

Again, $\Lambda_n(u,\{u_i\})$ and the Bethe equations have been written in their symmetric form.

4.2.3. The (F^-, G^-) case. Here we have

$$\Omega_1(u) = \mathrm{e}^{2u}\frac{\sinh(2u+6\eta)}{\sinh(2u+2\eta)}\frac{\cosh\left(u+\frac{5}{2}\eta\right)}{\cosh\left(u+\frac{9}{2}\eta\right)}\frac{\sinh\left(u+\frac{1}{2}\eta\right)}{\sinh\left(u+\frac{3}{2}\eta\right)}$$
$$\Omega_2(u) = -\frac{\sinh(2u+6\eta)}{\sinh(2u+4\eta)}\frac{\cosh(u+\frac{5}{2}\eta)}{\cosh\left(u+\frac{9}{2}\eta\right)} \tag{4.22}$$
$$\Omega_3(u) = \mathrm{e}^{-2u-4\eta}\frac{\cosh\left(u+\frac{3}{2}\eta\right)}{\cosh\left(u+\frac{9}{2}\eta\right)}$$

$$\mathcal{X}_1(u) = \mathrm{e}^{-2u}\frac{\cosh\left(u+\frac{3}{2}\eta\right)}{\cosh\left(u-\frac{3}{2}\eta\right)}\frac{x_1^{2N}(u)}{f^N(u)}$$

$$\mathcal{X}_2(u) = \frac{\sinh(2u)}{\sinh(2u+2\eta)}\frac{\cosh\left(u+\frac{1}{2}\eta\right)}{\cosh\left(u-\frac{3}{2}\eta\right)}\frac{x_2^{2N}(u)}{f^N(u)}$$

$$\mathcal{X}_3(u) = \mathrm{e}^{2u+4\eta}\frac{\sinh(2u)}{\sinh(2u+4\eta)}\frac{\cosh\left(u+\frac{1}{2}\eta\right)}{\cosh\left(u-\frac{3}{2}\eta\right)}\frac{\sinh\left(u+\frac{5}{2}\eta\right)}{\sinh\left(u+\frac{3}{2}\eta\right)}\frac{x_3^{2N}(u)}{f^N(u)} \tag{4.23}$$

$$\Theta(u_i) = \mathrm{e}^{-2u_i}\frac{\sinh(2u_i+2\eta)}{\sinh(2u_i)}\frac{\cosh\left(u_i+\frac{1}{2}\eta\right)}{\cosh\left(u_i+\frac{3}{2}\eta\right)}. \tag{4.24}$$

Therefore, the n-particle state $\Psi_n(\{u_i\})$ is an eigenfuction of the transfer matrix (3.8) for the $osp(2|1)$ vertex model with boundaries (F^-, G^-). The corresponding eigenvalue is given by

$$\begin{aligned}\Lambda_n(u,\{u_i\}) = {}& \Omega_1(u)\mathcal{X}_1(u)\prod_{i=1}^{n}\frac{\sin(u+u_i-\eta)}{\sinh(u+u_i+\eta)}\frac{\sinh(u-u_i-\eta)}{\sinh(u-u_i+\eta)}\\ &+\Omega_2(u)\mathcal{X}_2(u)\prod_{i=1}^{n}\frac{\sin(u+u_i+3\eta)}{\sinh(u+u_i+2\eta)}\frac{\sin(u-u_i+3\eta)}{\sinh(u-u_i+\eta)}\\ &\times\frac{\sinh(u+u_i)}{\sinh(u+u_i+\eta)}\frac{\sinh(u-u_i)}{\sinh(u-u_i+2\eta)}\\ &+\Omega_3(u)\mathcal{X}_3(u)\prod_{i=1}^{n}\frac{\sinh(u+u_i+4\eta)}{\sinh(u+u_i+2\eta)}\frac{\sinh(u-u_i+4\eta)}{\sinh(u-u_i+2\eta)}\end{aligned} \tag{4.25}$$

and the BA equations are now given by

$$\begin{aligned}\left(\frac{\sinh(u_i+\eta)}{\sinh(u_i-\eta)}\right)^{2N} = {}& \left(\frac{\cosh\left(u_i-\frac{1}{2}\eta\right)}{\cosh\left(u_i+\frac{1}{2}\eta\right)}\right)^2\prod_{\{j\neq i\}=1}^{n}\frac{\sin(u_i+u_j+2\eta)}{\sinh(u_i+u_j+\eta)}\frac{\sin(u_i-u_j+2\eta)}{\sinh(u_i-u_j-2\eta)}\\ &\times\frac{\sinh(u_i+u_j-\eta)}{\sin(u_i+u_j-2\eta)}\frac{\sinh(u_i-u_j-\eta)}{\sinh(u_i-u_j+\eta)}.\end{aligned} \tag{4.26}$$

Here, both $\Lambda_n(u,\{u_i\})$ and the BA equation were written with $u_i \to u_i - \eta$.

These three cases were also considered in [29] via the coordinate BA.

5. Conclusion

Here, with the aid of previous works [25, 26], the boundary algebraic BA was derived for two of the three-state graded 19-vertex models using a generalization of Tarasov's approach [37]. From our results for the transfer matrix, one can in principle derive the free-energy thermodynamics, the quasi-particle excitations behaviour, and the classes of universality governing the criticality of gapless regimes with integrable boundary conditions. Moreover, the rather universal formula we obtained for the eigenvectors could be useful in future computations of off-shell properties such as form factors and correlation functions with boundary conditions of relevant operators.

The algebraic BA for n-state models with periodic boundary conditions was developed by Martins in [38]. In a recent paper Galleas and Martins [39] have presented the algebraic BA for the vertex models based on superalgebras. Therefore we believe that the Martins's approach can be generalized to include the diagonal open boundary conditions.

Finally we observe that the vertex models discussed in this paper share a common algebraic structure with the non-graded 19-vertex models, the Zamolodchikov-Fateev [27] and the

Izergin-Korepin [28] models. Thus, we expected that a fusion procedure for the $sl(2|1)^{(2)}$ model as well as the analytical BA formulation based in the quantum group invariance of the $osp(2|1)$ model can reproduce our results.

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References

[1] Korepin V E and Essler F H L (ed) 1994 *Exactly Solvable Models of Strongly Correlated Electrons* (Singapore: World Scientific)
[2] Foerster A and Karowiski M 1993 *Nucl. Phys.* B **396** 611
[3] Foeerster A and Karowiski M 1993 *Nucl. Phys.* B **408** 512
[4] Bariev R Z, Klümpler A and Zittartz J 1995 *Europhys. Lett.* **32** 85
[5] Bedurfig G and Frahm H 1995 *J. Phys. A: Math. Gen.* **28** 4453
[6] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (London: Academic)
[7] Kulish P P and Sklyanin E K 1980 *Zap. Nauchn. Semin.* LOMI **95** 129
[8] Deguchi T, Fujii A and Ito K 1990 *Phys. Lett.* B **238** 242
[9] Bracken A J, Delius G W, Gould M D and Zhang Y-Z 1994 *J. Phys. A: Math. Gen* **27** 6551
[10] Maassarani Z 1995 *J. Phys. A: Math. Gen.* **28** 1305
[11] Gould M D, Links J R, Zhang Y-Z and Tsonhantjis I 1997 *J. Phys. A: Math. Gen.* **30** 4313
[12] Guan X-W, Foerster A, Grimm U, Römer R A and Schreiber M 2001 *Nucl. Phys.* B **618** 650
[13] Cherednik I V 1984 *Theor. Math. Phys.* **61** 977
[14] Sklyanin E K 1988 *J. Phys A: Math. Gen.* **21** 2375
[15] Mezincescu L and Nepomechie R I 1991 *J. Phys. A: Math. Gen.* **24** L17
[16] Mezincescu L and Nepomechie R I 1995 *J. Phys. A: Math. Gen.* **28** 5131
[17] deVega H J and Gonzales-Ruiz A 1993 *J. Phys. A: Math. Gen.* **26** L519
[18] deVega H J and Gonzales-Ruiz A 1994 *Nucl. Phys.* B **417** 553
[19] Yung C M and Batchelor M T 1995 *Nucl. Phys.* B **435** 430
[20] Faddeev L D and Takhtajan L A 1979 *Usp. Mat. Nauk.* **34** 13
[21] Bethe H A 1931 *Z. Physik* **71** 205
[22] Bracken A J, Ge X-Y, Zhang Y-Z and Zhou H-Q 1998 *Nucl. Phys.* B **516** 388
[23] Zhang Y-Z and Zhou H-Q 1998 *Phys. Lett.* A **244** 427
[24] Arnaudon D 1999 *J. High Energy Phys.* **12** 6
[25] Guang-Liang Li, Kang-Jie Shi and Rui-Hong Yue 2003 *Nucl. Phys.* B **670** 401
[26] Kurak V and Lima-Santos A 2004 *Nucl. Phys.* B **699** 595 (*Preprint* nlin.SI/0406050)
[27] Zamolodchikov A B and Fateev A V 1980 *Sov. J. Nucl. Phys.* **32** 298
[28] Izergin A G and Korepin V E 1981 *Commun. Math. Phys.* **79** 303
[29] Fireman E C, Lima-Santos A and Utiel W 2002 *Nucl. Phys.* B **626** 435 (*Preprint* nlin.SI/0110048)
[30] Zhou H-Q, Ge X-Y, Links J R and Gould M D 1999 *Nucl. Phys.* B **546** 779 (*Preprint* cond-mat/9809056)
[31] Fan H M and Wadati W 2001 *Nucl. Phys.* B **599** 561 (*Preprint* cond-mat/0008429)
[32] Mezincescu L and Nepomechie R I 1992 *Int. J. Mod. Phys.* **A 7** 5657 (*Preprint* hep-th/9206047)
[33] Kulish P P and Sklyanin E K 1982 *J. Sov. Math.* **19** 1596
[34] Bazhanov V V and Schadrikov A G 1989 *Theor. Math. Phys.* **73** 1302
[35] Mezincescu L R I and Nepomechie R I 1991 *Int. J. Mod. Phys.* **A 6** 5231
[36] Lima-Santos A 1999 *Nucl. Phys.* **B 558** 637 (*Preprint* solv-int/9906003)
[37] Tarasov V O 1988 *Teor. Math. Phys.* **76** 793
[38] Martins M J 1995 *Nucl. Phys.* B **450** 768 (*Preprint* hep-th/9502133)
[39] Galleas W and Martins M J 2004 *Nucl. Phys.* **B 699** 455 (*Preprint* nlin.SI/0406003)